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INVESTIGACIÓN

Free network adjustment: Minimum inner constraints and Pseudo-inverse approaches

Ajuste de red libre: Enfoques de condiciones internas mínimas y pseudo inversa

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ABSTRACT:

The least squares technique is a classic procedure to compute the coordinates of a geodetic network. Different approaches of this method have been developed to perform the least squares adjustment and thus solve the linearized system that relates the observations (internal geometry) and the reference system (external geometry). The free adjustment is a model that does not use fix coordinates in the design matrix, thus the solution does not have connection with referential system or datum. Therefore, the rank deficiency problem or datum defect, which in terms of linear algebra defines a singular matrix in the system of normal equations, must be solved. Two mainly approaches of free adjustment are used to solve a geodetic network, the minimum inner constraints and pseudo-inverse technique. Both models provide results in an arbitrary reference system, therefore, the S-transformation is a typical procedure to transform the result to a known datum. This paper presents a review of both methods and the necessary methodology to perform a free network adjustment. Finally, an example was presented to analyze the equivalence between both methods. The results obtained were compared with an estimation realized through the constrained adjustment.

Keywords: Least squares, Free adjustment networks, Minimum inner constrains, Pseudo-inverse, S- transformation.

RESUMEN

El método de los mínimos cuadrados es un procedimiento clásico para calcular las coordenadas de una red geodésica. Se pueden utilizar diferentes modelos para realizar el ajuste por mínimos cuadrados y así resolver el sistema linealizado que relaciona las observaciones (geometría interna) y el sistema de referencia (geometría externa). Uno de los métodos es el ajuste libre, el cual es un modelo que no utiliza coordenadas fijas en la matriz de diseño, por lo que la solución no tiene conexión con el sistema de referencia o datum. Por lo tanto, el problema de la deficiencia de rango o datum en términos de alegra lineal define una matriz singular para el sistema de ecuaciones normales que tiene que ser resuelto para ajustar una red geodésica. Mediante este método se utilizan principalmente dos enfoques de ajuste libre, la técnica de restricción mínima interna y la técnica pseudo inversa. Ambos modelos proporcionan resultados en un sistema de referencia arbitrario, por lo que la S-transformación es un procedimiento típico para transformar los resultados a un datum o sistema de referencia conocido. En este trabajo se presenta una revisión de ambos métodos y la metodología necesaria para realizar un ajuste de red libre. Finalmente se presentó un ejemplo para analizar la equivalencia entre ambos métodos. Los resultados obtenidos se compararon con una estimación realizada a través del modelo de ajuste con constreñimientos.

Palabras clave: Mínimos cuadrados, ajuste libre de redes, restricción mínima interna, Pseudo-inversa, Transformación S.

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Introduction

The main technique for coordinates computation of a geodetic network is the least squares method. This method relates the internal geometry (observations) and the external geometry (parameters). An important step on the least square adjustment of networks is the definition of a referential system or datum that allows connecting the internal geometry with a reference system. An approach for the definition of the datum is the selection of control points belonging to the external geometry network, so these points are considered fix in the design matrix during the adjustment procedure (absolute constrains) (Mikhail & Ackermann 1976, Mikhail & Gracie 1981, Teunissen 2011, Gemael et al. 2015, Ghilani 2018, Ogundare 2019). For the network adjustment process, the stability of the control points that define the datum is relevant, because displacements in them or changing in their positions can generate influences on the coordinate comparison or in deformation network analyses. Thus, the selected control points must have a good stability.

There are alternative procedures to reduce the influence of the stability of the control points among them we find the free adjustment. A main characteristic of the free network adjustment is not to consider the influence of external factors, therefore the errors associate to the control points are not considered (Mälzer et al. 1979, Blaha 1982, 1982a, Papo 1985, Even-Tzur 2011, Even-Tzur 2015). Thus, the stability and consistency problems of the coordinates that define the datum do not affect the adjustment results (Even-Tzur 2006). This characteristic is useful for geodetic monitoring activity, where the deformation elements can be estimated only if the control points that define the datum do not change between the measurement epochs (Even-Tzur 2011). In general, the network is treated as free network when all stations are assumed as unstable, and hence a minimum trace datum is used trough of free adjustment (Setan 2001).

The absences of datum parameters in the adjustment procedure generates the datum defect problem. For network adjustment by least square procedure this situation means that inversion of the normal equation matrix (N) cannot be computed by traditional techniques. Thus, the adjustment solution can be obtained by specific methods. Perelmuter (1979), Papo & Perlmutter (1981), Teunissen (1981), Leick (1982) and Ogundare (2019), present the free adjustment for networks using "minimum inner constraints", where are fixed a minimum quantity of approximate coordinates that permit the datum definition for 1D, 2D, and 3D networks. This coordinates are added to design matrix A, thus de rank deficient is solved and the inversion of the normal equation matrix is possible. Rao (1972), Mittermayer (1972), Grafarend & Schaffrin (1974), Perelmuter (1979), Teunissen (1981), Meissl (1982) and Ogundare (2019), present another approach with generalized inverses, in particular the Moore-Penrose inverse, this method provides a mathematical solution to inversion of normal equation matrix. For Ogundare (2019), in the context of network adjustment by least squares the minimal inner constraints method provides similar results that Pseudo-inverse. The goal of this work is to present a review of both methods with the main characteristics and their application in a geodetic network.

Least square estimation

The least square estimation provides a solution for an equation system with redundancy measurements through of a mathematical model. Particularly, for geodetic applications, the solution of these systems provides the parameters, mainly coordinates and heights (Vanicek & Wells 1972, Mikhail & Ackermann 1976, Mikhail & Gracie 1981, Cross 1990, Krakiwsky 1994, Vanicek 1995, Strang & Borre 1997, Wells & Krakiwsky 1997, Camargo 2000, Nievergelt 2000, Aduol 2003, Teunissen 2011, Brinker & Minnick 2013, Gemael *et al*. 2015, Ghilani 2018, Ogundare 2019, Schaffrin & Snow 2019). The basic functional model is presented in Equation 1:

$$y_{mx1} = A_{mxn} \cdot dx_{nx1} \qquad (1)$$

Where y corresponds to the observation vector of *dx* dimension *(m x 1)*, A is the design matrix *(m x n)* and is the unknown parameters vector *(n x 1).* The *y* vector is composed of surveying or geodetic measurements; therefore, this vector is contaminated by errors arising from the measurement's procedure. Thus, in order to reduce these errors on the results, the observation data is greater than the number of unknown parameters *(m > n)*. This condition, called redundancy, makes that system to be inconsistent and the unknown parameters can be estimated by different techniques. The network adjustment is the common geodetic procedure where the data of observations is redundant, therefore, the parameter estimation or adjustment process is necessary, the least squares solution is the main technique used for the network adjustment (Mikhail & Ackermann 1976, Mikhail & Gracie 1981, Teunissen 2011, Gemael *et al*. 2015, Ghilani 2018, O gundare 2019).

The least square solution is given by $dx = N^{-1} \cdot U$, where $N = (A^t \cdot W \cdot A)$ (normal equations) and $U = A^t \cdot W \cdot y$ (vector terms), W is the weight matrix (*m x m*). Therefore, the inversion of the N matrix is possible only if its determinant is different to zero (|N|≠0). Thus, the non-singular condition of N matrix means than the columns on the A matrix are not linearly dependent (Welsch 1979, Caspary *et al* 1987, Deakin 2005, Teunissen 2006, Ogundare 2019).

Geodetic network datum

For a geodetic network, the datum is defined as the parameters (coordinates) that permit the positioning of the network in an arbitrary referential system (Kuang 1996, Strang & Borre 1997, Ogundare 2019). In other words, the coordinates define the rotation, translation and scale of the system. The number of coordinates necessary to the definition and their dimensionality depended on the network type (1D, 2D, 3D) and the geodetic observables. For the observables, each of them can define rotation, translation or scale. The Table 1 presents the mainly geodetic observables and the datum element that define.

For the network type, Ghilani (2018), explains that to a 1D-network one vertical control point provides the datum definition (vertical translation). In addition for 2D and 3D classical networks, one control point (translation matrix) with same dimensionality of the network and one direction or azimuth (rotation matrix) are necessary, in both cases the scale is provided by the EDM sensor (observable). A particular case is the GNSS network, where the definition is done by one point, because the coordinates *x,y,z* provide the translation, the baseline components *dx, dy, dz* the orientations and scale (Ogundare 2019). Different sets of network configuration and parameters to define the survey geodetic network datum are presented in the Table 2.

According to the number of parameters that define the geodetic network we found two kinds of datums, over-constrained and minimum constrained. The over-constrained definition (more points than necessary to datum definition), provides a connection with referential system, that is an advantage. Conversely, the main problem for over-constraints definition is a stability and accuracy of controls points because the network accuracy can be affected by strains in the network geometry.

On the other hand, the minimum constrained datum is a solution without external influences. Therefore, the measurements or observations define the network geometry. A disadvantage is the absence of control points, this means in relative position for the coordinates. (Caspary *et al.*, 1987, Kuang 1996, Ogundare 2019).

Free adjustment

For the geodetic network, the internal geometry that is defined by observations of distances, directions or heights differences needs to be connected to a geodetic reference frame. For this, the external geometry composed of control coordinates are part of the least square adjustment process, commonly these coordinates are called constraints or fix parameters. This process permits to connect the observations with a geodetic reference frame (Deakin 2005, Teunissen 2006, Shahar & Even-Tzur 2014). For Deakin (2005) and Teunissen (2006), the observations provide partial definitions of a geodetic datum; therefore, the datum definition is done when the constraints parameters are used in the adjustment process.

The concept of free adjustment of geodetic networks is defined as the absence of fixed parameters in the adjustment process, in other words, there is no set of coordinates of the external geometry of the network during adjustment. Therefore, the elements of the internal geometry (observations) do not integrate the frame of reference during adjustment (Mittermayer 1972, Mälzer *et al.* 1979, Papo 1985, Deakin 2005, Teunissen 2006, Shahar & Even-Tzur 2014). The absences of datum parameters in the adjustment procedure generates the datum defect problem or rank deficient. In network adjustment by least square procedure this situation means that inversion of the normal equation matrix (N) cannot be obtained by traditional inverse procedure, because the matrix is singular, that is, the matrix has columns that are linear combination of the others. Two methods to compute the free adjustment network prevail: the minimal constrained and free adjustment through of generalized inverses. Both methods provide a solution to inversion of the normal equation matrix.

Minimum inner constraints model

The minimum inner constraints model incorporates a minimal amount of parameters necessary to define a referential system. Thus, the external geometry is not considered in

Table 1: Observations that define datum parameters, Adapted from Kuang (1996)

Observable	Translation (t)	Rotation (ω)	Scale (s)
Distances	-	-	s
Horizontal directions	-	-	-
Azimuth	-	ω-Z	-
Zenith directions	-	ω-X, ω-Y	-
GNSS/ Position	t-X, t-Y, t-Z	ω-X, ω-Y, ω-Z	s
2D position differences	-	ω-Z	s
Height differences	-	ω-Y, ω-Z	s

Table 2: Datum parameters, Adapted from Kuang (1996)

Network dimension	Observation type(s)	Network name	Datum parameters		
			Translation	Rotation	Scale
1	Height differences	Level network	1	--	--
2	Distances	Trilateration	$\begin{matrix}1 & 0\\ 0 & 1\end{matrix}$	$\begin{matrix}y_i^0\\ -x_i^0\end{matrix}$	--
2	Angles	Triangulation	$\begin{matrix}1 & 0\\ 0 & 1\end{matrix}$	$\begin{matrix}y_i^0\\ -x_i^0\end{matrix}$	$\begin{matrix}y_i^0\\ x_i^0\end{matrix}$
3	Distance / Angles	3D network	$\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}$	$\begin{matrix}0 & -z_i^0 & y_i^0\\ z_i^0 & 0 & -x_i^0\\ -y_i^0 & x_i^0 & 0\end{matrix}$	--

* The rotation matrix correspond to the vector representation of rotations (Zeng *et al.*, 2015)

the adjustment procedure, this model is called minimum constraints model. Therefore, the shape and the geometric size of the network is defined only by the internal geometry (Mikhail & Ackermann 1976, Mikhail & Gracie 1981, Snow 2002, Teunissen 2011, Ogundare 2019, Ghilani 2018). The normal equations (N) and the independent vector terms (U) to minimum inner constraints adjustment are presented following:

$$N = A^t \cdot W \cdot A + G \cdot G^t \qquad (2)$$

$$U = A^t \cdot W \cdot l \qquad (3)$$

In the normal equation N, the term $G \cdot G^t$ is added. The G matrix called constrained matrix span the null space of A and contains the inner datum parameters that define the dimensionality of the network. The configuration of the G matrix considers the rotation, translation and scale. Koch (1985), Setan (1995), Kuang (1996), Acar (2006), Rossikopoulos *et al.* (2016), Kotsakis (2018) and Ogundare (2019) presented the set of the G matrix for 3D network (Equation 4)

$$G^t = \begin{bmatrix} 1 & 0 & 0 & : & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & : & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & : & 0 & 0 & 1 & \cdots & 1 \\ 0 & Z_1 & -Y_1 & : & 0 & Z_2 & -Y_2 & \cdots & -Y_n \\ -Z_1 & 0 & X_1 & : & -Z_2 & 0 & X_2 & \cdots & X_n \\ Y_1 & -X_1 & 0 & : & Y_2 & -X_2 & 0 & \cdots & 0 \\ X_1 & Y_1 & Z_1 & : & X_2 & Y_2 & Z_2 & \cdots & Z_n \end{bmatrix} \qquad (4)$$

The first three rows of the G matrix correspond to the 3D translations, the next three row correspond to the 3D rotations and the last row to the scale parameters. For both cases X_1, Y_1, Z_1 are approximate coordinates of the network. The number of columns of the G matrix is equal to the number of parameters to be estimated. Thus, for a level network with five parameters to estimate, the translation of vertical component is expressed in the G matrix as $G = [1\ 1\ 1\ 1\ 1]^t$ (Setan, 1995). The adjustment parameters are presented in the Equation 5:

$$dx = N^{-1} \cdot A^t \cdot W \cdot A \cdot N^{-1} \cdot U \qquad (5)$$

The variance covariance matrix is:

$$Q_x = N^{-1} \cdot A^t \cdot W \cdot A \cdot N^{-1} \qquad (6)$$

Generalized inverses

In free network adjustment, the inversion of the normal equation (N) matrix can be computed by the generalized inverses (Rao 1972, Grafarend *et al.* 1974, Mälzer 1979, Leick 1982, Meissl 1982). In particular, The Moore-Penrose inverse is the main inverse used in geodetic networks problems called "Minimum norm least squares g-inverse" (Welsch 1979). Thus, for $A \in \mathbb{R}^{m\times n}$ and the linear system $A \cdot \vec{x} = \vec{y}$ with $\vec{x} \in \mathbb{R}^n$; $\vec{y} \in \mathbb{R}^m$. The Moore-Penrose pseudo-inverse provides a solution $\vec{x} = A^{\dagger}\vec{y}$, where $A^{\dagger}$ is a pseudo inverse of A. This matrix is unique and has the following properties (Equation 7):

$$\begin{aligned} &a. A \cdot A^{\dagger} \cdot A = A \\ &b. A^{\dagger} \cdot A \cdot A^{\dagger} = A^{\dagger} \\ &c. (A \cdot A^{\dagger})^t = A \cdot A^{\dagger} \\ &d. (A^{\dagger} \cdot A)^t = A^{\dagger} \cdot A \end{aligned} \qquad (7)$$

For full rank matrices (rows or columns linearly independent) the pseudo inverse can be obtained for non-square matrix. Therefore, if $m < n$ (rows linearly independent), $A^{\dagger} = A^t \cdot (A \cdot A^t)^{-1}$ and for $m > n$ (columns linearly independent), $A^{\dagger} = (A^t \cdot A)^{-1} \cdot A^t$. When $= n$, $A^{\dagger} = A^{-1}$. For matrices with deficient rank, the solution is commonly obtained by the Singular Value decomposition (SVD). Where A can be decomposed as $A = U\Sigma V^t$ and (Burdick, 2010).

The solution for least squares is

$$dx = N^{\dagger} \cdot A^{t} \cdot W \cdot y \qquad (8)$$

While the variance-covariance matrix is given by:

$$Q_x = N^{\dagger} \qquad (9)$$

S-transformation

The datum independence on the free network adjustment turns necessary the transformation of results of each epoch to a common datum for the particular analysis as deformation or densification, also by defects in the network configuration or practical limitations (such as obstruction of the line of sight or destruction of points) (Setan 1995, Setan & Singh 2001). Thus, the S-transformation technique permits the datum re-definition between referential systems or epochs (Baarda 1981, Gründig *et al*. 1985, Caspary *et al*. 1987, Setan 1995, Setan & Singh 2001, Teunissen 2006, Acar *et al*. 2008, Doganalp *et al*. 2010, Even-Tzur 2012). For the S-transformation, the estimation of parameters (dx) and the cofactor matrix Qx are necessary (Baarda 1981, Gründig *et al*. 1985, Caspary *et al*. 1987, Erol *et al*. 2006, Teunissen 2006, Acar *et al*. 2008, Doganalp *et al*. 2010, Guo 2012, Even-Tzur 2012, Schmitt 2013). The equations for the transformation are presented:

$$\begin{gathered} x_j = S_j \cdot x_i \\ Q_{xj} = S_j \cdot Q_{xj} \cdot S_j^t \\ S_j = (I - G' \cdot (G'^t \cdot I_J \cdot G')^{-1} \cdot G'^t \cdot I_j \end{gathered} \qquad (10)$$

Where:

x_j: Transformed parameters between referential system

Q_{xj}: Transformed cofactor matrix between referential system

S_j: Corresponds to the transformation matrix

I_j: Corresponds the diagonal matrix for defining the base after S-transformation, the diagonal elements can be one for elements that participate into datum definition or zero for other points

I: Identity matrix

G': Corresponds to the inner constraint matrix; this matrix is composed by rotation, translation and scale.

For a level network composed of four points, the S-transformation can be explained through an example. For this, we considered the transformation between two referential systems (Caspary *et al*. 1987, Setan 1995):

Ordinary minimum constraints with station 1 chosen as the datum point

Minimum trace where all stations are used for datum definition

The G matrix can be defined by scale constraint, therefore $G = [1\ 1\ 1\ 1]^t$ and for case (a) $I_{ja} = [1\ 0\ 0\ 0]^t$ and for case (b) $I_{jb} = [1\ 1\ 1\ 1]^t$. The identity matrix has a dimension of 4x4. Thus, the transformation from (a) to (b) is defined by:

$$\begin{gathered} x_{ja} = S_{jb} \cdot x_i \\ S_{jb} = (I - G \cdot (G^t \cdot I_{Jb} \cdot G)^{-1} \cdot G^t \cdot I_{jb} \end{gathered} \qquad (11)$$

Application

As an example, the free adjustment was applied in the downstream geodetic network of Salto Caxias hydroelectric power station located in the Parana state, Brazil (Figure 1). The external geometry of this network have four (4) stations while the internal geometry is composed of six (6) distances, twelve (12) angles and one (1) azimuth observation (Table 3) (Granemann, 2005).

The minimum inner constraints method and pseudo-inverse approach were applied in network adjustment according to section 5 and 6 respectively. Additionally the constrained adjustment was calculated with the P1 point as fixed and oriented to point P3 (90º). The stochastic model of the observations corresponds to measures of variability (standard deviation), therefore the weight matrix was defined by the inverse of variance of the observations.

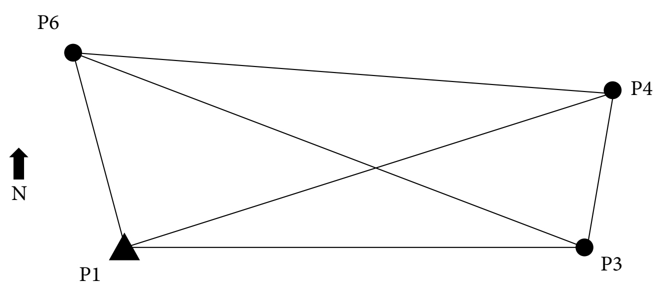

Figure 1: 2D network of Salto Caxias

The results for the constrained adjustment are presented in the Table 4, in this case, the point P1 is the control point or absolute constraint.

For the minimum inner constraints method the G matrix has a dimension of 2x8 and is composed only of translation parameters:

$$G^t = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \qquad (12)$$

For free adjustment procedure applied to nonlinear models, the determination of the initial coordinates and iterative process are a critical step. Tsutomu (1986) & Katsumi (1990) related the influence of the determination of initial coordinates and the free adjustment results. Kotsakis (2012) explains the relation between the stability of the

Table 3: Network observation of downstream geodetic network of Salto Caxias hydroelectric power station

Network observations								
Line	**Distance (m)**	**σ (mm)**	**Angle**	**Value**	**σ ('')**	**Angle**	**Value**	**σ ('')**
P1 - P3	232.809	1.0	P3 P1 P4	75° 49' 39.36''	1.4	P4 P6 P3	18° 17' 10.68''	1.5
P6 - P4	653.555	4.0	P4 P1 P3	17° 6' 24.84''	1.7	P4 P6 P1	83° 58' 5.88''	2.0
P4 - P3	205.711	3.0	P3 P1 P6	267° 03' 55.44''	1.2	P1 P4 P6	20° 12' 16.2''	2.8
P3 - P1	581.863	3.0	P1 P6 P4	276° 01' 53.76''	2.0	P3 P4 P1	56° 17' 1.32''	1.8
P1 - P4	670.340	4.0	P6 P4 P3	283° 30' 42.1''	2.2	P1 P3 P6	21° 23' 0.6''	0.9
P6 - P3	637.678	3.0	P4 P3 P1	253° 23' 29.7''	1.1	P6 P3 P4	85° 13' 28.92''	0.8
			Az P1 P3	90° 0'0''	1.0			

Table 4: Constrained adjustment results

Point	**Constrained least squares**			
	X (m)	**σ (m)**	**Y (m)**	**σ (m)**
P1	1000.000	Control point	1000.000	Control point
P3	1581.8635	0.0018	1000.0000	0.0001
P4	1640.6799	0.0017	1197.1894	0.0019
P6	988.0811	0.0009	1232.5038	0.0009

network and the iterative convergent solution. Thus, the free adjustment applied to networks with nonlinear models should have a special treatment to represent the network geometry. In this work, we used the procedure presented by Tsutomu (1986), therefore the initial coordinates correspond to the adjustment coordinates obtained by the constrained adjustment of the same network with the point P3 fixed.

In Figure 2 from Kotsakis (2012), AB line is a distance, the grey line represents the probable positions of point B with respect to point A and the red line represents the possible positions of B with respect to the reference system. The B point is the real location of this coordinate. B1 and B2 points are two different initial coordinates for iterative process. In this case, the network stability is better if the B1 is the initial coordinate due to the proximity between the coordinates B1 and B. For Ipsen (2011), this situation can be explained due to B1 or B2 is far from B. the method may not converge. This means the solution does not represent the network geometry. In other words, for rank- deficient models the convergent solution is not necessarily unique.

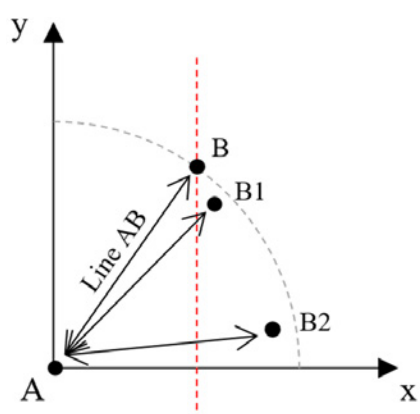

Figure 2: Adapted from Kotsakis (2012), position of initial coordinates to free adjustment

The adjustment parameters and their precisions for both approaches are presented in Table 5:

The global test (chi-square) was applied to each adjustment, the results are presented in Table 6 to confidence level of 95% with (n-u)=(19-8)=11 degrees of freedom for free adjustments and (n-u)=(19-6)=13 degrees of freedom for the constrained adjustment.

Table 5: Least square solution by Pseudo-inverse approach and Minimum inner constraints method

Point	Pseudo inverse approach				Minimum inner constraints			
	X (m)	σ (m)	Y (m)	σ (m)	X (m)	σ (m)	Y (m)	σ (m)
P1	999.9963	0.0011	999.9966	0.0016	999.9963	0.0009	999.9966	0.0005
P3	1581.8598	0.0010	999.9966	0.0015	1581.8598	0.0009	999.9966	0.0005
P4	1640.6762	0.0011	1197.1861	0.0022	1640.6762	0.0009	1197.1861	0.0014
P6	988.0775	0.0011	1232.5005	0.0017	988.0775	0.0009	1232.5005	0.0008

The S-transformation was applied to both adjustment processes, so one control point was selected (P1) and considered as constrained parameter, its coordinates are (1000.00m, 1000.00m). The measured bearings and the scale by the measured distances define the orientation of the network. The dimension of the G' and I matrix is 2x8. Both matrix are presented following:

$$G^t = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \quad (13)$$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (14)$$

The adjustment vector parameters transformed through S-Transformation is obtained through:

$$\hat{x} = x_o + x_j \quad (15)$$

Where x_0 corresponds to initial vector of the parameters, the variance covariance matrix was obtained with equation 5. The results and their precision are presented in the Table 7:

Finally, differences between the free adjustments and constrained adjustment are summarized in the Table 8.

Table 6: Global test to each adjustment approach

Method	Estimate	Critical value (95%)	Status
Pseudo inverse	6.1648	19.67510	Pass
Minimum inner constraints	6.1648	19.67510	Pass
Parametric adjustment	6.9851	24.73560	Pass

Table 7: S-transformation results to Pseudo-inverse approach and Minimum inner constraints

Point	Pseudo inverse approach				Minimum inner constraints			
	X (m)	σ (m)	Y (m)	σ (m)	X (m)	σ (m)	Y (m)	σ (m)
P1	1000.000	0.0000	1000.000	0.0000	1000.000	0.0000	1000.000	0.0000
P3	1581.8635	0.0018	1000.000	0.0028	1581.8635	0.0018	1000.000	0.0000
P4	1640.6799	0.0020	1197.1894	0.0036	1640.6799	0.0017	1197.1894	0.0019
P6	988.0811	0.0015	1232.5038	0.0009	988.0811	0.0009	1232.5038	0.0009

Table 8: Summary of differences between free adjustments and constrained adjustment

Method / adjustment elements	Pseudo-inverse	Minimum inner constraints	Constrained adjustment
Coordinate unknowns	8	8	6
Datum defect	X,Y Coordinates and orientation	X,Y Coordinates and orientation	X,Y Coordinates and orientation
Datum definition	Free	Free	Fix
Degrees of freedom	11	11	13
Posterior variance	0.5604	0.5604	0.5373
χ^2 estimate	6.1648	6.1648	6.9851
Critical value of χ^2	19.6751	19.6751	24.7356
Global test (one – tailed)	Pass	Pass	Pass

Conclusions

Two approaches for free adjustment computations were presented, the pseudo-inverse method that provides a mathematical solution to compute the inverse of normal equation matrix (N) and therefore maintains the classical formulation to the least squares. On the other hand, the minimum inners constraints method needs the addition of the G matrix, which contains the minimum parameters to datum definition. Thus, the G matrix spans null space to the design matrix A, consequently the lack of information of the network datum or the rank deficiency is solved and the inversion of the normal equation matrix is done.

For the example presented, both methods provide equivalent results for the parameters, and global test, therefore according to Ogundare (2019) it was verified the similarity of both methods. The S-transformation is necessary to transform datum from the arbitrary referential system (provided by the free adjustment) to the reference datum. The results of the S-transformation have equivalent results for both methods.

One of the main differences between both methods (free and parametric) is related to the definition of the network geometry, which in the case of free adjustment is obtained without the need to set coordinates in the design matrix. Therefore, the network geometry is defined in an arbitrary system from the observations themselves. This feature is useful for evaluating the quality of a network adjustment.

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